Using rank characters to decompose convex persistence modules

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University of Iowa Algebra Seminar

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Summary of results

- We give a definition of the rank of a higher-dimensional persistence module in terms of convex modules.
- This generalizes the definition in the one-dimensional case and resembles a virtual character.
- These convex modules play the role of basis elements and thus can distinguish between two direct sums of convex modules.
- This rank invariant produces a one-sided test for determining if a persistence module decomposes into a direct sum of convex modules.
- When it does, this gives an algorithm for producing its barcode completely in terms of elementary linear algebra.

What is persistent homology?

An algebraic method used in topological data analysis, meaning an algebraic means to measure topological features of shapes, data, and functions.

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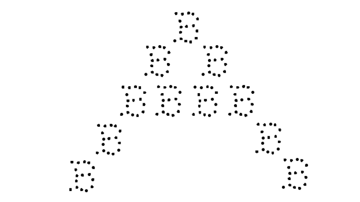

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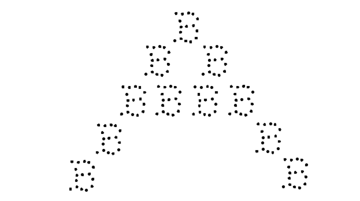

As one increases a threshold, at what scale do we observe changes in some representation of the data?

Applications

Where is it used?

- Analysis of social and spatial networks
- Coverage and hole detection of wireless sensor fields
- Shape classification
- Medical imagery and prediction
- Signal denoising
- Pattern detection software
- Statistics
- Machine learning
- Dynamical systems

Setup

Consider a nested sequence (filtration) of topological spaces:

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

We want to

1. Understand the topological structure of each individual space X_i
2. Understand how these topological features persist throughout the nested sequence.

How do we measure persistence?

Homology!

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Homology!

Every filtration

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

of topological spaces produces a sequence of homology groups (vector spaces)

$$H_*(X_0) \to H_*(X_1) \to H_*(X_2) \to \cdots$$

This sequence is a **persistence module.**

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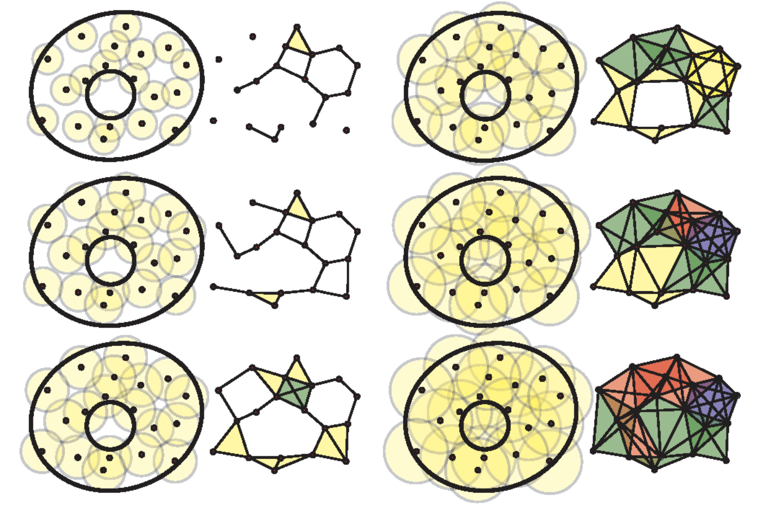

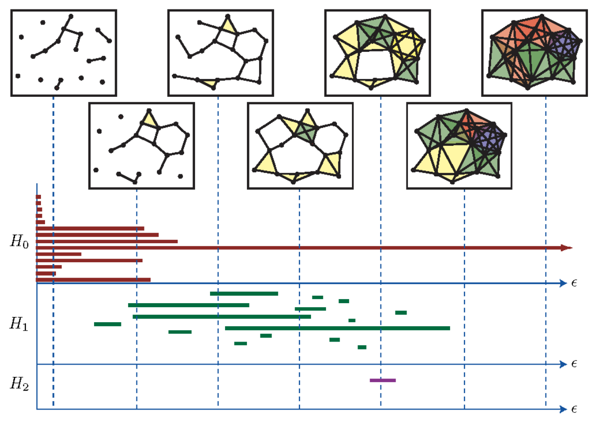

A **barcode** is a visual representation of $H_i(M; k)$. The horizontal axis corresponds to the parameter while the vertical axis corresponds to an ordering of the homology generators.

Theorem (Zomorodian, Carlsson (2005))

The rank of $H_*(M_*^i; k)$ equals the number of intervals which contain i.

A barcode is then the persistent analogue of a Betti number.

One-dimensional case

Consider a subposet $P \subseteq \mathbb{R}$. A **one-parameter persistence module** is a functor $M : P \to \mathrm{Vec}_k$.

But this is nothing other than a poset representation!

Define the rank map

$$\mathrm{Rank}\, M(s,t) = \mathrm{Rank}[M(s) \to M(t)],\ \ s \leq t \in P$$

This rank map uniquely describes M and its decomposition into interval modules:

$$\mathrm{Rank}\, M = \mathrm{Rank}\left(\bigoplus_I k_I\right) \Rightarrow M = \bigoplus_I k_I$$

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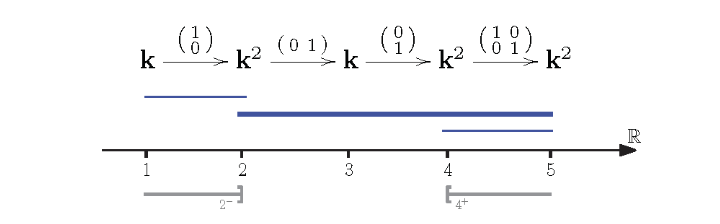
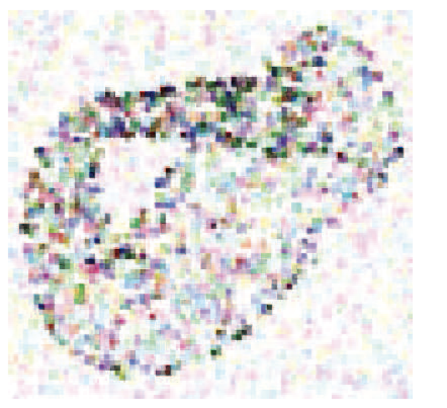
Reinterpreted, one-dimensional persistence modules are just representations of equioriented $\overrightarrow{A_n}$:



Higher dimensional case

- What is the proper generalization of a persistence barcode to higher dimensions?
- How should we define rank in this case?
- What “basis” elements should we consider?



Algebraically, persistence modules are nothing more than representations of posets.

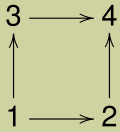
While many different posets have a physical interpretation in the literature, of particular interest are convex (or interval) persistence modules, which correspond to those modules which can be uniquely determined from their support.

Definition

For a poset P, a nonempty subset I is called **convex** if whenever $s \leq u \leq t$ and $s, t \in I$, then $u \in I$.

Example (Convex subsets)

Poset P:



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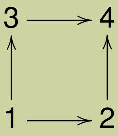
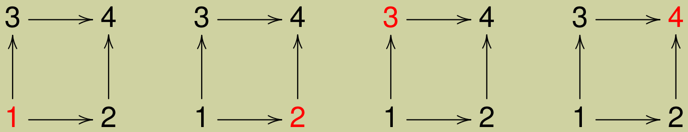


Convex subsets: Singletons:



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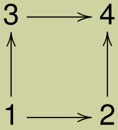
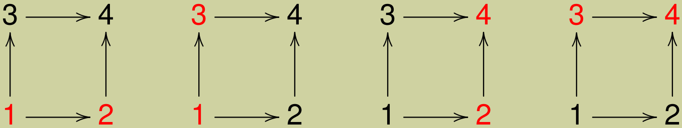


Convex subsets: Intervals



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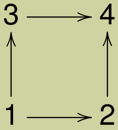
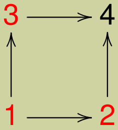
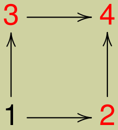


Convex subsets: Two intervals





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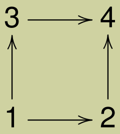
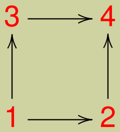


Convex subsets: The poset



For any (finite, connected) poset P, define the set

$$\Sigma := \{S \subseteq P \mid S \text{ is connected and convex}\}.$$

Each $S \in \Sigma$ defines a module M_S by

$$M_S(p) := \begin{cases} k & \text{if } p \in S \\ 0 & \text{if } p \notin S \end{cases} \quad \Leftrightarrow \quad M_S([a,b]) := \begin{cases} 1 & \text{if } \{a,b\} \subseteq S \\ 0 & \text{if } \{a,b\} \not\subseteq S \end{cases}$$

Each set S gives rise to an algebra $A_S = \sum [a_i, b_i]$, where the sum is over the maximal intervals contained in S. That is,

$$A_S = \sum_{[a_i,b_i] \subseteq T_S} [a_i, b_i],$$

$$T_S := \{[a,b] \mid [a,b] \subseteq S \text{ and } [a,b] \subseteq [c,d] \subseteq S \Rightarrow [a,b] = [c,d]\}.$$

Alternatively, $A_S = kQ_P/I$.

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Example (Rectangle)

$$\begin{array}{ccc} 3 & \longrightarrow & 4 \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$$

For $S = \{1, 2, 3\}$, we get M_S and A_S to be, respectively,

$$\begin{array}{ccc} k & \xrightarrow{0} & 0 \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle 0} \\ k & \xrightarrow[1]{} & k \end{array}$$

$$[1, 2] + [1, 3]$$

Example (Rectangle)

$$\begin{array}{ccc} 3 & \longrightarrow & 4 \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$$

For $S' = \{1, 2, 3, 4\}$, we get $M_{S'}$ and $A_{S'}$ to be, respectively,

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$$[1, 4]$$

Example (Rectangle, cont.)

$$M_S(A_S) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad M_S(A_{S'}) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}.$$

$\mathrm{Rank}(M_S(A_S)) = 1$ and $\mathrm{Rank}(M_S(A_{S'})) = 0$. For this P, we have the following convex modules and the algebras generated by them:

	1	2	3	4	5	6
S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$
A_S	$[1] = [1,1]$	$[2]$	$[3]$	$[4]$	$[1,2]$	$[1,3]$

	7	8	9	10	11
S	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
A_S	$[2,4]$	$[3,4]$	$[1,2]+[1,3]$	$[2,4]+[3,4]$	$[1,4]$

Example (Rectangle, cont.)

$(\mathrm{Rank}(M_S(A_{S'})))_{S,S'\in\Sigma}$

	1	2	3	4	12	13	24	34	123	234	1234
1	1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0
12	1	1	0	0	1	0	0	0	1	0	0
13	1	0	1	0	0	1	0	0	1	0	0
24	0	1	0	1	0	0	1	0	0	1	0
34	0	0	1	1	0	0	0	1	0	1	0
123	1	1	1	0	1	1	0	0	1	0	0
234	0	1	1	1	0	0	1	1	0	1	0
1234	1	1	1	1	1	1	1	1	1	1	1

Example ($\overrightarrow{A}_n$)

$$A_n : 1 \to 2 \to \cdots \to n-1 \to n$$

The maximal intervals agree with the indecomposable representations:

$$[i, j] : 0 \to \cdots \to 0 \to \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{R} \to 0 \to \cdots \to 0,$$

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In this case, $\Sigma = \bigcup_{1 \leq k \leq n} L_k$, where L_k is the set of all such maximal intervals of width k. We order Σ by $i < j \implies L_i < L_j$ and lexicographically within each L_k.

$$M_S(A_{S'})_{b,a} = \begin{cases} 1 & \text{if } [a,b] \middle| S' \text{ and } [a,b] \subseteq S \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{[x,y]}(A_{[z,w]})_{b,a} = \begin{cases} 1 & \text{if } x \leq a = z \leq b = w \leq y \\ 0 & \text{otherwise} \end{cases}$$

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Example ($\overrightarrow{A}_n$, cont.)

$$\text{Rank}(M_{[x,y]}(A_{[z,w]})) = \begin{cases} 1 & \text{if } x \leq z \leq w \leq y \\ 0 & \text{otherwise} \end{cases}$$

Suppose $S \in L_i$ and $S' \in L_j$.

- $i < j \Rightarrow \text{Rank}(M_S(A_{S'})) = 0$
- $i = j \Rightarrow \text{Rank}(M_S(A_{S'})) = 1 \Leftrightarrow S = S'$
- $j < i \Rightarrow \text{Rank}(M_S(A_{S'})) = 1$ if $S' \subseteq S$, $= 0$ otherwise

Thus, the rank matrix $\text{Rank}(M_S(A_{S'}))_{S',S \in \Sigma}$ is a lower triangular block matrix with diagonal blocks $I_n, \ldots, I_1$, and so is invertible.

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Example ($\overrightarrow{A}_n$, cont.)

For instance, consider $A_4 : 1 \to 2 \to 3 \to 4$. Then

$$\Sigma = \{[1], [2], [3], [4], [1,2], [2,3], [3,4], [1,3], [2,4], [1,4]\}.$$

For $S = [1,2]$ and $S' = [2,4]$,

$$M_S(A_S)_{b,a} = \begin{cases} 1 & \text{if } a = 1,\ b = 2 \\ 0 & \text{otherwise} \end{cases} \qquad M_S(A_{S'})_{b,a} = 0 \ \ \forall a, b.$$

$$M_S(A_S) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_S(A_{S'}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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4	0	0	0	1	0	0	0	0	0	0
12	1	1	0	0	1	0	0	0	0	0
23	0	1	1	0	0	1	0	0	0	0
34	0	0	1	1	0	0	1	0	0	0
13	1	0	1	0	1	1	0	1	0	0
24	0	1	0	1	0	1	1	0	1	0
14	1	1	1	1	1	1	1	1	1	1

Theorem (-, Meyer (2021))

Let P be a finite, connected poset and let Σ be the set of connected, convex subsets of P. Then the rank matrix $(\mathrm{Rank}(M_S(A_{S'})))_{(S,S')\in\Sigma^2}$ is invertible.

This says that a finite set of elements of the algebra given by convex modules is sufficient to separate direct sums of convex modules, *i.e.*, two direct sums of convex modules that agree on them are equal:

If $M_1 = \bigoplus_M \bigoplus_{j=1}^{m_M} M_j$ and $N_1 = \bigoplus_N \bigoplus_{j=1}^{n_N} N_j$ agree on all A_S for each S convex, then $M_1 \cong N_1$.

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Important takeaways:

- When M_S is a direct sum of convex modules, multiplying the rank vector of M_S by the inverse of this matrix produces the multiplicities of the convex modules in the direct sum decomposition. That is, this describes its decomposition into its barcode.
- Note that this is only a one-sided test! Rank is too coarse of a description to determine if a module is *not* a direct sum of convex modules.

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- Note that this is only a one-sided test! Rank is too coarse of a description to determine if a module is *not* a direct sum of convex modules.

Example

Poset= $\overrightarrow{A}_3 : 1 \to 2 \to 3$

Representation: $\mathbb{R}^3 \xrightarrow{(1\ 1\ 0)} \mathbb{R} \xrightarrow{(0\ 1)^T} \mathbb{R}^2$

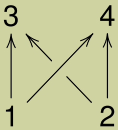
Maximal convex subintervals:
$\Sigma = \{[1],[2],[3],[1,2],[2,3],[1,2,3]\}$

Rank vector: $\mathbf{v} = (3\ 1\ 2\ 1\ 1\ 1)^T$

$(\text{Rank}(M_S(A_{S'})))^{-1}_{(S,S')\in\Sigma^2}\mathbf{v} = (2\ 0\ 1\ 0\ 0\ 1)^T$

Thus, the representation's barcode/decomposition is $[1]^2 \oplus [3] \oplus [1,3]$.

Example



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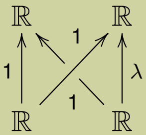
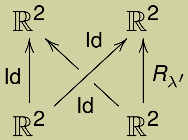
23	0	1	1	0	0	0	1	0	1	0	0	1	1
24	0	1	0	1	0	0	0	1	0	1	0	1	1
123	1	1	1	0	1	0	1	0	1	0	1	1	1
124	1	1	0	1	0	1	0	1	0	1	1	1	1
134	1	0	1	1	1	1	0	0	1	1	1	0	1
234	0	1	1	1	0	0	1	1	1	1	0	1	1
1234	1	1	1	1	1	1	1	1	1	1	1	1	1

Example

$\lambda, \lambda' \in \mathbb{R}$ and $R_{\lambda'} = \begin{pmatrix} 1 & \lambda' \\ 0 & 1 \end{pmatrix}$



M_λ



$M'_{\lambda'}$

M_λ is an indecomposable thin yet is not a direct sum of convex modules for all $\lambda \neq 1$.

$M'_{\lambda'}$ is indecomposable whenever $\lambda' \neq 0$ and has the same rank vector for all $\lambda' \in \mathbb{R}$, yet $M'_{\lambda'}$ is not a direct sum of convex modules except in the case $\lambda' = 0$, in which case $M'_0 \cong M_1 \oplus M_1$.

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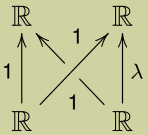

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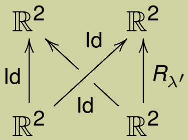

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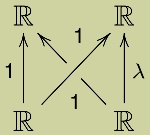

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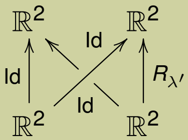

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Future Questions

- How exactly does our definition of rank relate to the definition of rank in the Botnan-Oppermann-Oudot paper?
- When a module which is not a direct sum of convex modules produces a "barcode" with all positive entries, does this have any special meaning? In particular, does this module have some relation to a module with the same barcode which is a direct sum of convex modules? Or is there a way to refine our definition of rank?

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Thank you!

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